Advanced Algorithms

October 21, 2025

Consider the following

Given a bipartite graph $G = (L \cup R, E)$, define the following Linear Program:

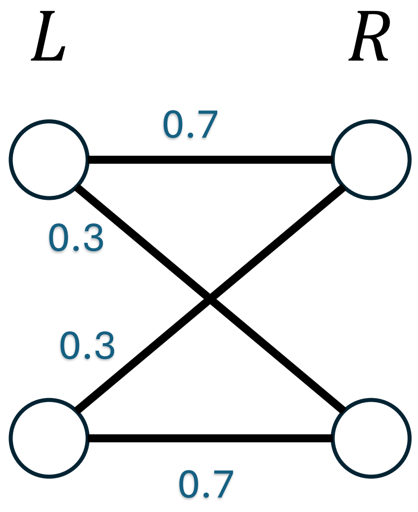
$$\min \sum_{e \in E} c_e x_e$$

such that:

$$\sum_{e \in \delta(u)} x_e = 1 \qquad \text{for all } u \in L$$

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \text{for all } v \in R$$

$$x_e \geq 0 \qquad \text{for all } e \in E$$



What are integer solutions to this LP? Think about what its dual would be.

Logistics

- Assignment due tonight, late deadline = Thursday
- Future assignments: will facilitate groups for those who want
- Academic integrity:
  - Working together is encouraged
  - Asking me for hints is encouraged
  - You **must** submit your own work

Where we are

1. Linear Prog. can model many interesting problems, and is in P

2. Integer Prog. can model MANY problems but is NP-hard

3. We can use Linear Prog. to understand certain Integer Programs

*Also, a *language* of optimization

Where we are

1. Linear Prog. can model many interesting problems, and is in P

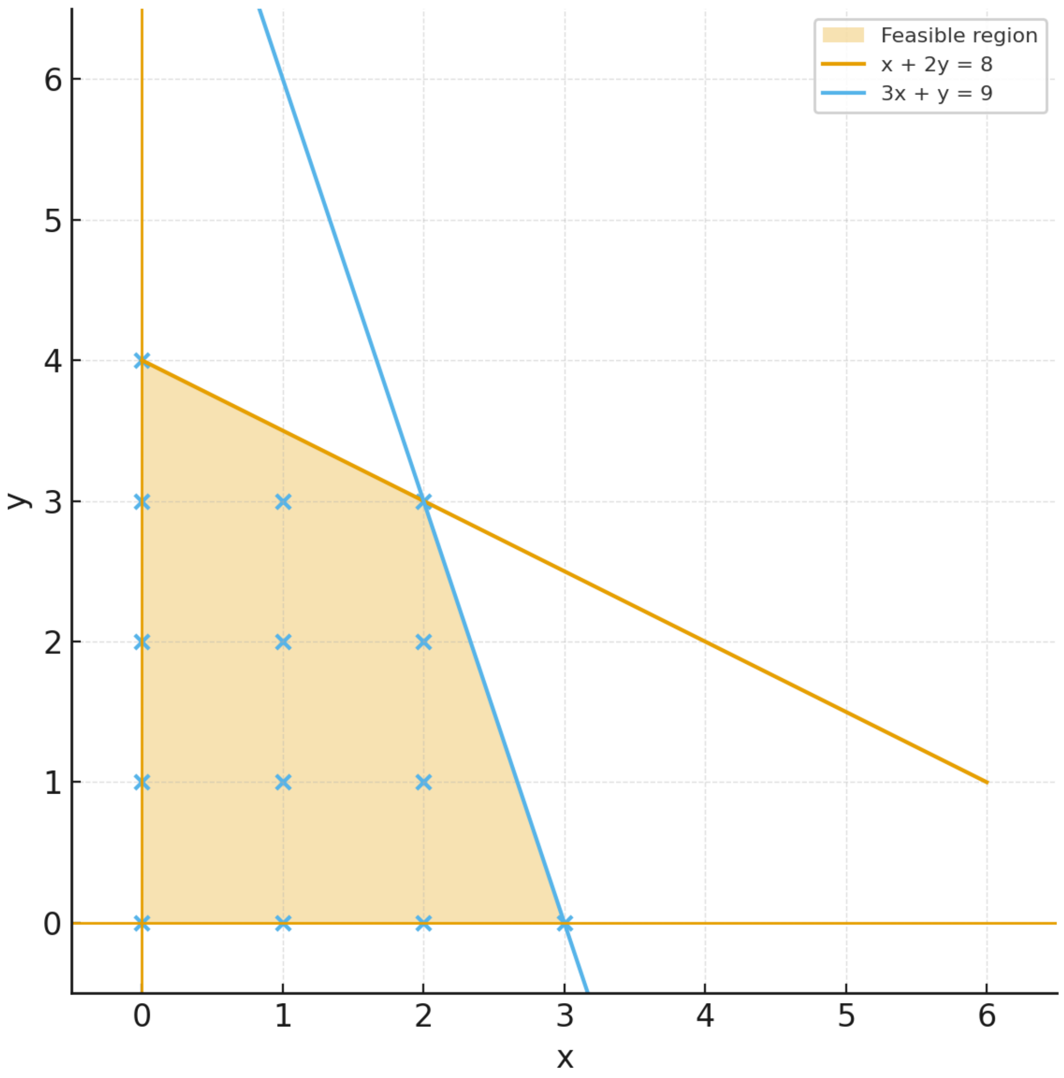
2. Integer Prog. can model MANY problems but is NP-hard

3. We can use Linear Prog. to understand certain Integer Programs

*Also, a *language* of optimization

Today

- Integer Programs we can solve efficiently, optimally



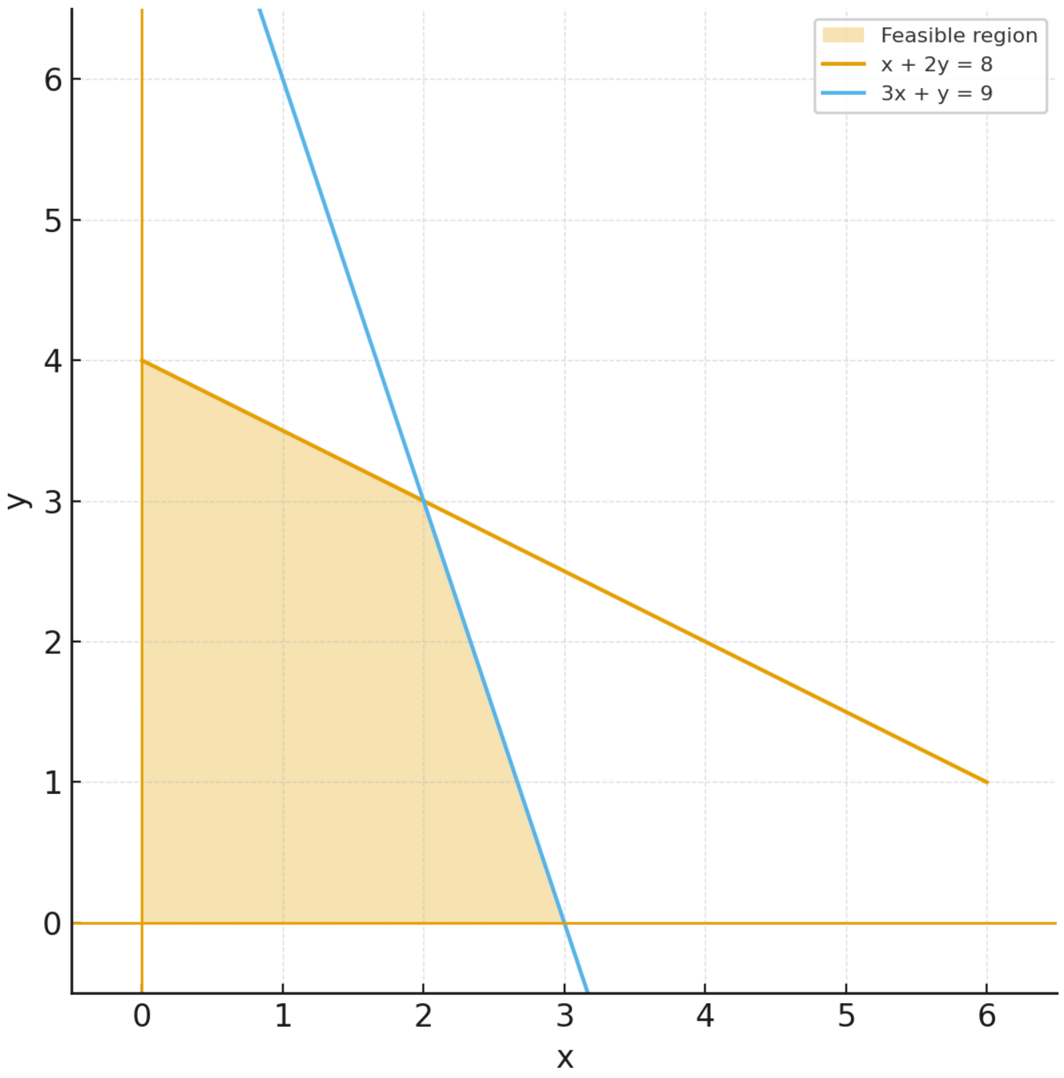
Today

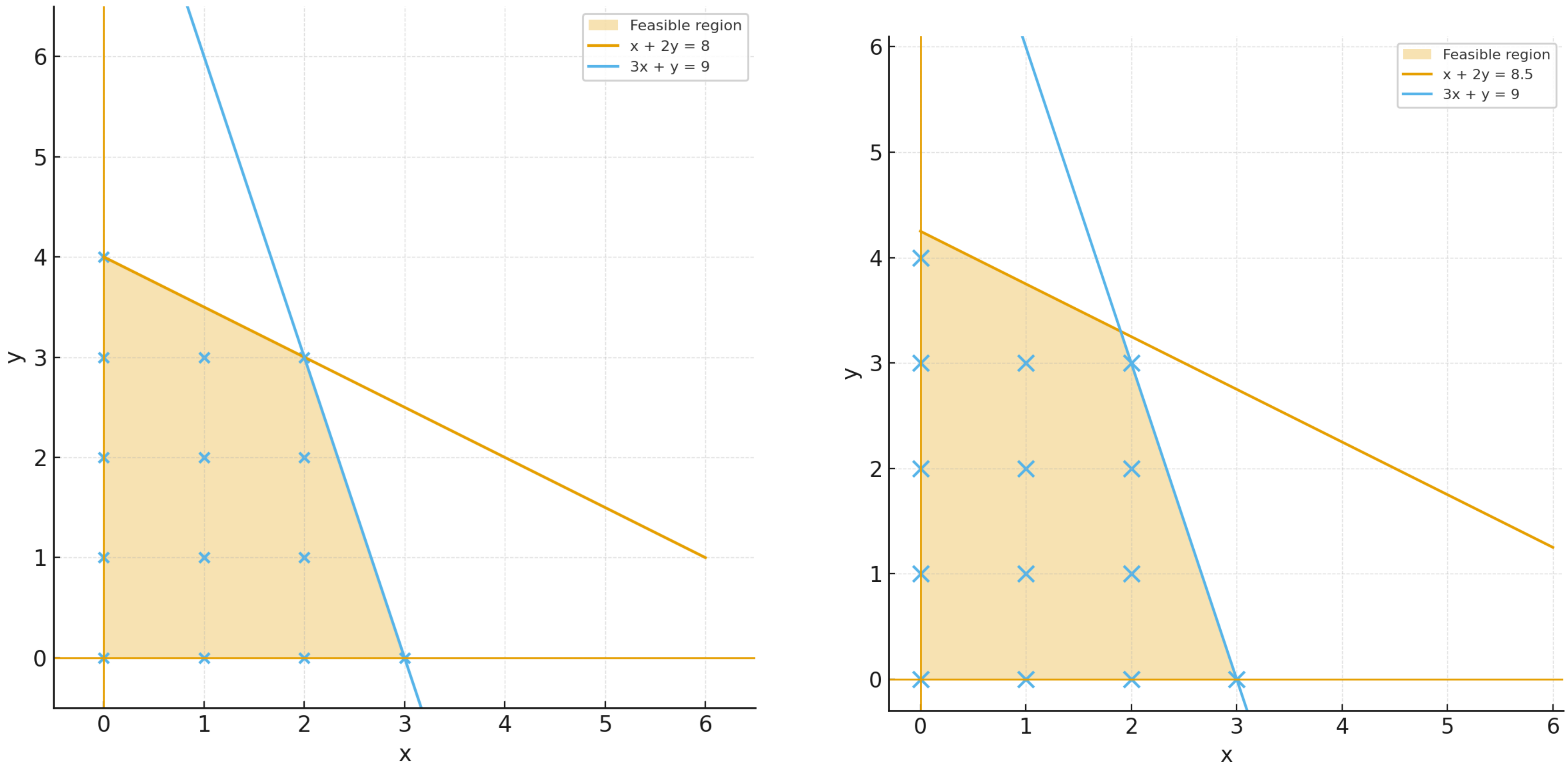
- Integer Programs we can solve efficiently, optimally

- The optimal solution to a linear program will always lie at a "corner" of the feasible region

- AKA an extreme point

- All corners integral:
  - A "perfect formulation"
  - Integrality gap is 1




Feasible region
x + 2y = 8
3x + y = 9
Feasible region
x + 2y = 8.5
3x + y = 9
x
y

Main Theorem

Given a bipartite graph $G = (L \cup R, E)$ define the following Linear Program:

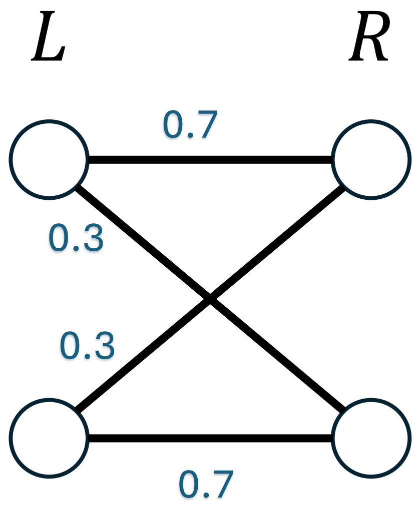
$$\min \sum_{e \in E} c_e x_e$$

such that:

$$\sum_{e \in \delta(u)} x_e = 1 \qquad \text{for all } u \in L$$

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \text{for all } v \in R$$

$$x_e \geq 0 \qquad \text{for all } e \in E$$



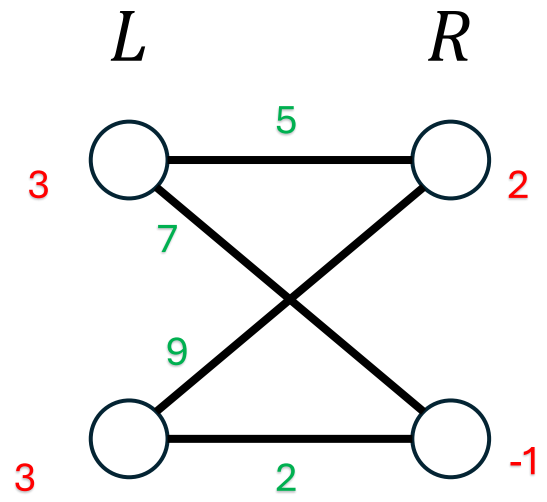
The above Linear Program is a **perfect formulation**.

Duality

$\max \ \sum_{u\in L} p_u + \sum_{v\in R} p_v$

such that:

$p_u + p_v \leq c_e$ for all $e = (u,v) \in E$



Seem familiar?

Duality

$\min \sum_{e \in E} c_e x_e$

such that:

$\sum_{e \in \delta(u)} = 1$ for all $u \in L$

$\sum_{e \in \delta(v)} = 1$ for all $v \in R$

$x_e \geq 0$ for all $e \in E$

$\max \sum_{u \in L} p_u + \sum_{v \in R} p_v$

such that:

$p_u + p_v \leq c_e$ for all $e = (u, v) \in E$

Where we are

1. Linear Prog. can model many interesting problems, and is in P

2. Integer Prog. can model MANY problems but is NP-hard

3. We can use Linear Prog. to understand certain Integer Programs

*Also, a *language* of optimization